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Guidelines and Recommendations to Construct Theories for Metallic and Composite Plates

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This work has evaluated the refinement of some classical theories, such as the Kirchhoff and Reissner–Mindlin theories, adding generalized displacement variables (up to fourth-order) to the Taylor-type expansion in the thickness plate direction. Isotropic, orthotropic, and laminated plates have been analyzed, varying the thickness ratio, orthotropic ratio, and stacking sequence of the layout. Higher-order theories have been implemented according to the compact scheme known as the Carrera unified formulation. The results have been restricted to simply-supported orthotropic plates subjected to harmonic distributions of transverse pressure for which closed-form solutions are available. For a given plate problem (isotropic, orthotropic, or laminated), the effectiveness of each employed generalized displacement variable has been established comparing the error obtained accounting for and removing the variable in the plate governing equations. A number of theories have therefore been constructed imposing a given error with respect to the available best results. Guidelines and recommendations that are focused on the proper selection of the displacement variables that have to be retained in refined plate theories are then furnished. It has been found that the terms that have to be used according to a given error vary from problem to problem, but they also vary when the variable that has to be evaluated (displacement, stress components) is changed. Diagrams (errors in terms of geometrical and orthotopic ratios) and graphical schemes have been built to establish the appropriate theories with respect to the data of the problem under consideration.

Nomenclature

A_k	=	integration domain in the z direction
$\boldsymbol{a}$	=	generic variable vector
a	=	dimension of the plate in the x direction
b	=	dimension of the plate in the y direction
$\boldsymbol{C}$	=	constitutive equations matrix
$\tilde{C}_{ij}$	=	material coefficients
$\tilde{\boldsymbol{C}}_{pp}, \tilde{\boldsymbol{C}}_{pn}, \tilde{\boldsymbol{C}}_{np}, \tilde{\boldsymbol{C}}_{nn}$	=	material stiffness subarrays
$\boldsymbol{D}_p, \boldsymbol{D}_n, \boldsymbol{D}_{n\Omega}, \boldsymbol{D}_{nz}$	=	differential operator matrices
E, E_L, E_T, E_z	=	Young's moduli
$E_{\tau s}, E_{\tau_{,z} s}, E_{\tau s_{,z}}, E_{\tau_{,z} s_{,z}}$	=	integrals in the z direction
F_s	=	thickness function of the variation
F_τ	=	thickness function of the variable
h	=	thickness of the plate
$\mathbf{I}_p, \mathbf{I}_{np}$	=	additional arrays to perform integration by parts
$\boldsymbol{K}_{uu}^{k\tau s}$	=	fundamental nuclei of stiffness matrix
k	=	layer index
L_e	=	external work
m, n	=	waves number in the x and y direction
N	=	order of the expansion in the thickness direction
N_l	=	number of layers
$\boldsymbol{P}_{u\tau}^k$	=	external load
p_z	=	bisinusoidal transverse distributed load
$\bar{p}_z$	=	amplitude of the bisinusoidal transverse distributed load
$\hat{U}_{x_\tau}^k, \hat{U}_{y_\tau}^k, \hat{U}_{z_\tau}^k$	=	amplitudes of the harmonic distributions of the displacement variables
$\boldsymbol{u}$	=	displacements vector
u_x, u_y, u_z	=	displacement components in the x, y, and z directions
$\bar{u}_z$	=	dimensionless displacement component
x, y, z	=	coordinates reference system
α, β	=	parameters for the harmonic distributions along x and y directions
Γ_k	=	edge of surface Ω_k
$\delta\boldsymbol{a}$	=	vector of the variation of a generic variable
$\delta_{u_z}, \bar{\delta}_{u_z}$	=	percentage variations of an output variable
$\boldsymbol{\epsilon}_p, \boldsymbol{\epsilon}_n$	=	in-plane and normal/transverse strains
ν, ν_{LT}, ν_{Lz}	=	Poisson's ratios
$\Pi_{uu}^{k\tau s}$	=	matrix to perform boundary conditions
$\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_n$	=	in-plane and normal/transverse stresses
$\bar{\sigma}_{xx}, \bar{\sigma}_{xz}, \bar{\sigma}_{zz}$	=	dimensionless stress components
Ω_k	=	in-plane integration domain

I. Introduction

MOST known theories pertaining to the analysis of beam-, plate- and shell-type structures originated from the intuition of some structural analysis pioneers. One of the first contributions to the theory of beam structures is due to the genius of Leonardo Da Vinci (see Ballarini [1]). He correctly established all the features of the displacement field of a beam, hypothesizing a linear distribution of the strain on the cross section, as can be seen in the right-to-left written document in Fig. 1 (see [1] for more details about this issue). Some other well-known contributions are those by Euler [2], Bernoulli [3], Cauchy [4], Poisson [5], Kirchhoff [6], De Saint-Venant [7], Love [8], Reissner [9], Mindlin [10], and Vlasov [11], among others. In most cases, these "axiomatic" intuitions lead to a simplified kinematics of the true three-dimensional deformation state of the considered structure: the section remains plane, the section/thickness deformation can be discarded, shear strains are negligible, etc.

For a complete review of this topic, including laminated composite structures, the readers can refer to the many available survey papers

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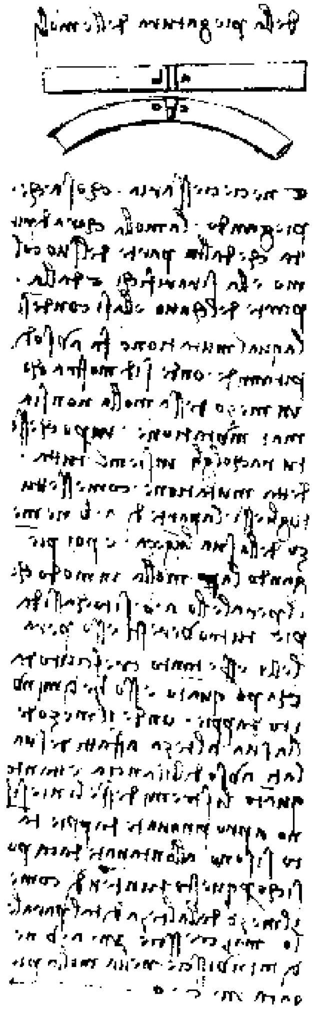


Fig. 1 Da Vinci's discussion of the deformation of a beam/spring with rectangular cross section [70].

on beams, plates and shells. Among these, excellent reviews have been made in the papers by Ambartsumian [12], Librescu and Reddy [13], Grigolyuk and Kulikov [14], Kapania and Raciti [15,16], Kapania [17], Noor et al. [18–20], Reddy and Robbins [21], Carrera [22,23], Qatu [24,25] as well as in the books by Librescu [26], Reddy [27], and Qatu [28]. These papers review theories that deal with layerwise models (LWMs), and equivalent single-layer models (ESLMs). According to Reddy [27], the number of displacement variables is kept independent of the number of constitutive layers in the ESLM, whereas the same variables are independent in each layer for LWM cases. Attention of the present work has been restricted to the bending analysis of plates, a review of contributions related to vibrations and buckling can be found in the papers by Noor et al. [29,30], Leissa [31], Ji and Waas [32].

As an alternative to the axiomatic approach, approximated theories have been introduced employing asymptotic-type expansions of unknown variables over the section (beam case) or thickness (plate/shell geometries). The order of magnitude of significant terms is evaluated referring to a geometrical parameter (thickness-to-length in the case of plates/shells, and section dimension to length in the case of beams). The asymptotic approach furnishes consistent approximations. This means that all the retained terms are those which have the same order of magnitude as the introduced perturbation parameter when the latter vanishes. Papers on the application of asymptotic methods to shell structures can be found in Cicala [33], Fettahlioglu and Steele [34], Berdichevsky et al. [35,36], Widera et al. [37,38], and Spencer et al. [39], as well as in the monographes by Cicala [40] and Gol'denweizer [41]. Works pertaining to asymptotic methods applied to beam structures have been drawn up by Volovoi et al. [42,43], Popescu et al. [44,45], and Yu et al. [46–48].

Both the axiomatic and the asymptotic methods have historically been motivated by the need to work with simplified theories that are capable of leading to simple formulas and equations which can be solved by hand calculation. Up to five decades ago, in fact, it was quite prohibitive to solve problems with many unknowns (more than 5, 6). Nowadays, this limitation no longer holds; the preceding mentioned pioneers would quite probably propose to extend their own methods and limitations; if they had owned a computer, they would have solved more complicated beam/plate/shell problems. Of course, the formulation of more complicated problems would be difficult without the introduction of appropriate techniques that are suitable for computer implementations.

The approach herein discussed makes use of such a suitable technique: enhanced beam/plate/shell formulations are, in fact, constructed using a suitable condensed notation techniques for computer implementations. This approach was introduced by the first author during the last decade and it is referred to as the Carrera unified formulation (CUF) for beams, plates and shell structures [22,49–54]. Governing equations, in both strong and weak form, are given in terms of a few fundamental nuclei whose form does not depend on either the order of the introduced approximations or on the choices made for the base functions in the thickness direction (for plates/shells) or over the section (for beams).

The present work deals with problems with only displacement variables formulated using the principle of virtual displacements (PVD). Related computer forms only use nine FORTRAN statements to implement a large variety of theories which permit the hierarchical evaluations of any type of approximations. This work is restricted to plate geometries and closed-form solutions.

In short, CUF makes it possible to implement those terms which had been neglected by the preceding cited pioneers. To obtain more general conclusions and to draw general guidelines and recommendation in building plate theories for metallic and composites plates, it would be of great interest to evaluate the effectiveness of each refined theory term. This has been done in the present paper. In CUF, in fact, the role of each displacement variable in the solution is investigated by measuring the loss of accuracy due to its being neglected. A term is considered ineffective, i.e., negligible, if it does not affect the accuracy of the solution with respect to a reference three-dimensional solution. Reduced kinematics models, based on a set of retained displacement variables, are then obtained for each considered configuration. Full and reduced models are then compared in order to highlight the sensitivity of a kinematics model to variations in the structural problem. This method can somehow be considered as a mixed axiomatic/asymptotic approach because it furnishes asymptoticlike results, starting from a preliminary axiomatic choice of the base functions. A companion investigation, related to a finite element analysis of beams made of isotropic materials, has been proposed in [55].

The paper is organized as follows. A brief description of the adopted formulation is furnished in Sec. II. The method used to evaluate the effectiveness of various plate theories is introduced in Sec. III. The addressed structural problems, together with the results and discussion, are provided in Sec. IV. The main conclusions are outlined in Sec. V.

II. Carrera Unified Formulation for Plates

The CUF is a technique which models a large variety of beam/plate/shell structures in an unified manner. Details of CUF can be found in the already mentioned papers. However, for the sake of completeness, CUF details which are relevant for the present work are herein reconsidered. According to CUF, the governing equations are written in terms of a few fundamental nuclei which do not formally depend on both the order of the expansion N used in the thickness direction, z, and the variables' description [layerwise (LW) or equivalent single-layer (ESL)]. In the case of ESL, which is herein considered, the displacement field for plates is modeled in the following manner:

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \qquad \tau = 0, 1, \ldots, N \tag{1}$$

where F_τ are functions of z. $\mathbf{u}_\tau$ is the displacements vector and N stands for the order of the expansion. According to Einstein's notation, the repeated subscript τ indicates summation. In this work, Taylor polynomials are used for the expansion:

$$F_\tau = z^\tau, \qquad \tau = 0, 1, \ldots, N \tag{2}$$

N is assumed to be as high as four. Therefore the displacement field is

$$\begin{aligned} u_x &= u_{x_1} + z u_{x_2} + z^2 u_{x_3} + z^3 u_{x_4} + z^4 u_{x_5} \\ u_y &= u_{y_1} + z u_{y_2} + z^2 u_{y_3} + z^3 u_{y_4} + z^4 u_{y_5} \\ u_z &= u_{z_1} + z u_{z_2} + z^2 u_{z_3} + z^3 u_{z_4} + z^4 u_{z_5} \end{aligned} \tag{3}$$

The Reissner–Mindlin plate model approximation [9,10] [also known as first-order shear deformation theory (FSDT), in the case of laminates] can be obtained acting on the F_τ expansion. Two conditions have to be imposed: 1) first-order approximation kinematics field and 2) displacement component u_z constant above the cross section, i.e. $u_{z_2} = 0$. The resultant displacement model is

$$u_x = u_{x_1} + z u_{x_2} \qquad u_y = u_{y_1} + z u_{y_2} \qquad u_z = u_{z_1} \tag{4}$$

The Kirchhoff-type approximation [6] [also known as classical laminate theory (CLT)] can also be obtained using a penalty technique for the shear correction factor. First-order models require the use of reduced material stiffness coefficients to correct the thickness locking (see [56,57]).

A. Geometric and Constitutive Relations

The adopted coordinate frame is presented in Fig. 2. a and b are the lengths of the plate along x and y axes, respectively. h is the thickness of the plate. The stresses, σ^k, and the strains, ϵ^k, of the k-th layer are grouped as follows:

$$\begin{aligned} \boldsymbol{\sigma}_p^k &= \{\, \sigma_{xx}^k \quad \sigma_{yy}^k \quad \sigma_{xy}^k \,\}^T, \qquad & \boldsymbol{\epsilon}_p^k &= \{\, \epsilon_{xx}^k \quad \epsilon_{yy}^k \quad \epsilon_{xy}^k \,\}^T \\ \boldsymbol{\sigma}_n^k &= \{\, \sigma_{xz}^k \quad \sigma_{yz}^k \quad \sigma_{zz}^k \,\}^T, \qquad & \boldsymbol{\epsilon}_n &= \{\, \epsilon_{xz}^k \quad \epsilon_{yz}^k \quad \epsilon_{zz}^k \,\}^T \end{aligned} \tag{5}$$

Subscript n is related to the in-plane components, whereas p refers to the out-of-plane components. In the case of linear theory, the strains-displacements relations are

$$\boldsymbol{\epsilon}_p^k = \boldsymbol{D}_p \mathbf{u}^k \qquad \boldsymbol{\epsilon}_n^k = \boldsymbol{D}_n \mathbf{u}^k = (\boldsymbol{D}_{n\Omega} + \boldsymbol{D}_{nz}) \mathbf{u}^k \tag{6}$$

with

$$\boldsymbol{D}_p = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \qquad \boldsymbol{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{D}_{nz} = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \tag{7}$$

In the case of orthotropic materials, Hooke's law holds:

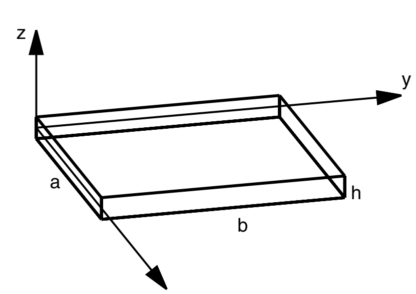


Fig. 2 Reference coordinate frame.

$$\boldsymbol{\sigma}^k = \boldsymbol{C}^k \boldsymbol{\epsilon}^k \tag{8}$$

According to Eqs. (5), the previous equation becomes

$$\boldsymbol{\sigma}_p^k = \tilde{\boldsymbol{C}}_{pp}^k \boldsymbol{\epsilon}_p^k + \tilde{\boldsymbol{C}}_{pn}^k \boldsymbol{\epsilon}_n^k \qquad \boldsymbol{\sigma}_n^k = \tilde{\boldsymbol{C}}_{np}^k \boldsymbol{\epsilon}_p^k + \tilde{\boldsymbol{C}}_{nn}^k \boldsymbol{\epsilon}_n^k \tag{9}$$

where matrices $\tilde{\boldsymbol{C}}_{pp}^k$, $\tilde{\boldsymbol{C}}_{nn}^k$, $\tilde{\boldsymbol{C}}_{pn}^k$ and $\tilde{\boldsymbol{C}}_{np}^k$ are

$$\tilde{\boldsymbol{C}}_{pp}^k = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{bmatrix}^k, \qquad \tilde{\boldsymbol{C}}_{nn}^k = \begin{bmatrix} \tilde{C}_{55} & \tilde{C}_{45} & 0 \\ \tilde{C}_{45} & \tilde{C}_{44} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix}^k$$
$$\tilde{\boldsymbol{C}}_{pn}^k = \tilde{\boldsymbol{C}}_{np}^{k^T} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & \tilde{C}_{36} \end{bmatrix}^k \tag{10}$$

For the sake of brevity, the dependence of the elastic coefficients $[\tilde{C}]_{ij}^k$ on Young's modulus, Poisson's ratio, the shear modulus and the fiber angle is not reported. It can be found in [58] or [27].

B. Governing Differential Equations

The governing equations are obtained via the PVD:

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \{\delta \epsilon_p^{k^T} \sigma_p^k + \delta \epsilon_n^{k^T} \sigma_n^k\} \, \mathrm{d}\Omega_k \, \mathrm{d}z = \sum_{k=1}^{N_l} \delta L_e^k \tag{11}$$

where the integration domains Ω_k and A_k indicate the reference plane of the lamina and its thickness, respectively. N_l stands for the number of layers. δL_e^k is the virtual variation of the external work which takes into account the external loadings for a generic layer k. Using Eqs. (6) and (9), Eq. (11) becomes

$$\begin{aligned} \int_{\Omega_k} \int_{A_k} &\{(\mathbf{D}_p \delta \mathbf{u}^k)^T [(\mathbf{C}_{pp}^k \mathbf{D}_p + \mathbf{C}_{pn}^k (\mathbf{D}_{np} + \mathbf{D}_{nz})) \mathbf{u}^k] \\ &+ ((\mathbf{D}_{np} + \mathbf{D}_{nz}) \delta \mathbf{u}^k)^T [(\mathbf{C}_{pn}^{k^T} \mathbf{D}_p + \mathbf{C}_{nn}^k (\mathbf{D}_{np} \\ &+ \mathbf{D}_{nz})) \mathbf{u}^k]\} \, \mathrm{d}\Omega_k \, \mathrm{d}z = \delta L_e^k \end{aligned} \tag{12}$$

Using Eq. (1), Eq. (12) is rewritten as follows:

$$\begin{aligned} \int_{\Omega_k} \int_{A_k} &\{[(\mathbf{D}_p)(F_s \delta \mathbf{u}_s^k)]^T [(\mathbf{C}_{pp}^k \mathbf{D}_p + \mathbf{C}_{pn}^k (\mathbf{D}_{np} + \mathbf{D}_{nz}))(F_\tau \mathbf{u}_\tau^k)] \\ &+ [(\mathbf{D}_{np} + \mathbf{D}_{nz})(F_s \delta \mathbf{u}_s^k)]^T [(\mathbf{C}_{np}^k \mathbf{D}_p + \mathbf{C}_{nn}^k (\mathbf{D}_{np} + \mathbf{D}_{nz})) \\ &\times (F_\tau \mathbf{u}_\tau^k)]\} \, \mathrm{d}\Omega_k \, \mathrm{d}z = \delta L_e^k \end{aligned} \tag{13}$$

The following notation is introduced:

$$(E_{\tau s}, E_{\tau_{,z} s}, E_{\tau s_{,z}}, E_{\tau_{,z} s_{,z}}) = \int_{A_k} (F_\tau F_s, F_{\tau_{,z}} F_s, F_\tau F_{s_{,z}}, F_{\tau_{,z}} F_{s_{,z}}) \, \mathrm{d}z \tag{14}$$

where the subscript z indicates partial derivative with respect to z. Equation (13) becomes

$$\begin{aligned} \int_{\Omega_k} &[(\mathbf{D}_p \delta \mathbf{u}_s^k)^T (E_{\tau s} \mathbf{C}_{pp}^k \mathbf{D}_p \mathbf{u}_\tau^k + E_{\tau s} \mathbf{C}_{pn}^k \mathbf{D}_{np} \mathbf{u}_\tau^k + E_{\tau_{,z} s} \mathbf{C}_{pn}^k \mathbf{u}_\tau^k) \\ &+ (\mathbf{D}_{np} \delta \mathbf{u}_s^k)^T (E_{\tau s r} \mathbf{C}_{np}^k \mathbf{D}_p \mathbf{u}_\tau^k + E_{\tau s} \mathbf{C}_{nn}^k \mathbf{D}_{np} \mathbf{u}_\tau^k + E_{\tau_{,z} s} \mathbf{C}_{nn}^k \mathbf{u}_\tau^k) \\ &+ (\delta \mathbf{u}_s^k)^T (E_{\tau s_{,z}} \mathbf{C}_{np}^k \mathbf{D}_p \mathbf{u}_\tau^k + E_{\tau s_{,z}} \mathbf{C}_{nn}^k \mathbf{D}_{np} \mathbf{u}_\tau^k \\ &+ E_{\tau_{,z} s_{,z}} \mathbf{C}_{nn}^k \mathbf{u}_\tau^k] \, \mathrm{d}\Omega_k = \delta L_e^k \end{aligned}$$

The integration by parts is requested to obtain strong form of the differential equations on Ω_k and boundary conditions on Γ_k. For a generic variable $\mathbf{a}^k$, the integration by parts states

$$\int_{\Omega_k} ((\mathbf{D}_\Omega)\delta\mathbf{a}^k)^T \mathbf{a}^k \, d\Omega_k = -\int_{\Omega_k} \delta\mathbf{a}^{k^T} ((\mathbf{D}_\Omega^T)\mathbf{a}^k) \, d\Omega_k + \int_{\Gamma_k} \delta\mathbf{a}^{k^T} ((\mathbf{I}_\Omega)\mathbf{a}^k) \, d\Gamma_k \tag{15}$$

where $\Omega = p, np$. The governing equations are

$$\delta\mathbf{u}_s^{k^T}: \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_\tau^k = \mathbf{P}_{u\tau}^k \tag{16}$$

with boundary conditions state

$$\Pi_{uu}^{k\tau s} u_\tau^k = \Pi_{uu}^{k\tau s} \bar{u}_\tau^k \tag{17}$$

$\mathbf{P}_{u\tau}^k$ is the external load in Eq. (16). The fundamental nucleus, $\mathbf{K}_{uu}^{k\tau s}$, is assembled through the depicted indexes, τ and s, which consider the order of the expansion in z for the displacements.

Superscript k denotes the assembly on the number of layers. The explicit form of the fundamental nuclei is

$$\begin{aligned}\mathbf{K}_{uu}^{k\tau s} = \{&(-\mathbf{D}_p^T)[\mathbf{C}_{pp}^k E_{\tau s}\mathbf{D}_p + \mathbf{C}_{pn}^k E_{\tau s}\mathbf{D}_{np} + \mathbf{C}_{pn}^k E_{\tau,_z s}] \\ &- (\mathbf{D}_{np}^T)[\mathbf{C}_{pn}^k E_{\tau s}\mathbf{D}_p + \mathbf{C}_{nn}^k E_{\tau s}\mathbf{D}_{np} + \mathbf{C}_{nn}^k E_{\tau,_z s}] \\ &+ [\mathbf{C}_{np}^k E_{\tau s,_z}\mathbf{D}_p + \mathbf{C}_{nn}^k E_{\tau s,_z}\mathbf{D}_{np} + \mathbf{C}_{nn}^k E_{\tau,_z s,_z}]\}\end{aligned} \tag{18}$$

and for boundary conditions

$$\begin{aligned}\Pi_{uu}^{k\tau s} = \{&(I_p)[C_{pp}^k E_{\tau s} D_p + C_{pn}^k E_{\tau s} D_{np} + C_{pn}^k E_{\tau,_z s}] \\ &+ (I_{np})[C_{pn}^k E_{\tau s} D_p + C_{nn}^k E_{\tau s} D_{np} + C_{nn}^k E_{\tau,_z s}]\}\end{aligned} \tag{19}$$

The following additional arrays have been introduced to perform the integration by parts:

$$\mathbf{I}_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}; \qquad \mathbf{I}_{np} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

C. Closed-Form Solutions

Navier-type closed-form solutions are possible for simply-supported plates made of orthotropic materials. The following properties hold:

$$C_{pp16} = C_{pp26} = C_{pn63} = C_{pn36} = C_{nn45} = 0$$

The following harmonic assumptions can be made for the displacement variables:

$$\begin{aligned} u_{x_\tau}^k &= \sum_{m,n} (\hat{U}_{x_\tau}^k) \cos\frac{m\pi x_k}{a_k} \sin\frac{n\pi y_k}{b_k} \qquad k = 1, N_l \\ u_{y_\tau}^k &= \sum_{m,n} (\hat{U}_{y_\tau}^k) \sin\frac{m\pi x_k}{a_k} \cos\frac{n\pi y_k}{b_k} \qquad \tau = 1, N \\ u_{z_\tau}^k &= \sum_{m,n} (\hat{U}_{z_\tau}^k) \sin\frac{m\pi x_k}{a_k} \sin\frac{n\pi y_k}{b_k} \end{aligned} \tag{21}$$

where $\hat{U}_{x_\tau}^k$, $\hat{U}_{y_\tau}^k$ and $\hat{U}_{z_\tau}^k$ are the amplitudes, m and n are the number of waves (they go from 0 to ∞) and a_k and b_k are the dimensions of the plate.

The governing equations in Eq. (16) become a system of algebraic equations. The explicit form of fundamental nuclei becomes

$$\begin{aligned} K_{uu_{11}} &= C_{55}E_{\tau,_z s,_z} + C_{11}\alpha^2 E_{\tau s} + C_{66}\beta^2 E_{\tau s} \\ K_{uu_{12}} &= C_{12}\alpha\beta E_{\tau s} + C_{66}\alpha\beta E_{\tau s} \\ K_{uu_{13}} &= -C_{13}\alpha E_{\tau,_z s} + C_{55}\alpha E_{\tau s,_z} \qquad K_{uu_{21}} = K_{uu_{12}} \\ K_{uu_{22}} &= C_{44}E_{\tau,_z s,_z} + C_{22}\beta^2 E_{\tau s} + C_{66}\alpha^2 E_{\tau s} \\ K_{uu_{23}} &= -C_{23}\beta E_{\tau,_z s} + C_{44}\beta E_{\tau s,_z} \\ K_{uu_{31}} &= C_{55}\alpha E_{\tau,_z s} - C_{13}\alpha E_{\tau s,_z} \qquad K_{uu_{32}} = C_{44}\beta E_{\tau,_z s} - C_{23}\beta E_{\tau s,_z} \\ K_{uu_{33}} &= C_{33}E_{\tau,_z s,_z} + C_{44}\beta^2 E_{\tau s} + C_{55}\alpha^2 E_{\tau s} \end{aligned} \tag{22}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$.

It should be noted that no assumptions on the approximation order have been made. It is therefore possible to obtain refined plate models without changing the formal expression of the nuclei components. This is the key-point of CUF which permits to implement any-order plate theories using only nine FORTRAN statements. An example of FORTRAN code statements is reported in the Appendix.

III. Method used to Evaluate the Effectiveness of Various Displacement Variables

Significant advantages are offered by refined plate theories in terms of accuracy of the solution and detection of nonclassical effects. The drawback of these theories is that a higher computational effort is necessary because of the presence of a larger number of displacement variables.

This work is an effort to understand the convenience of using a fully refined model rather than a reduced one each time a plate problem has to be dealt with. The effectiveness of each term, as well as the terms that have to be retained in the formulation, are investigated as follows:

1) The problem data are fixed (geometry, boundary conditions, loadings, materials and layer layouts).

2) A set of output variables is chosen (maximum displacements, stress/displacement component at a given point, etc.).

3) A theory is fixed, that is, the terms that have to be considered in the expansion of u_x, u_y, and u_z are established (not only the order, e.g., a linear case neglecting constant terms could be implemented): all the possibilities are considered (see the examples next, in particular those in Fig. 3).

4) A reference solution is used to establish the accuracy (the $N = 4$ case is assumed as the best-reference result because it offers an excellent agreement with the three-dimensional solutions).

5) CUF is used to generate the governing equations for the considered theories.

6) The effectiveness of each term is numerically established measuring the error produced compared with the reference solution.

7) Any term which does not give any contribution to the mechanical response is not considered as effective in the kinematics model.

8) The most suitable kinematics model is then detected for a given structural layout.

A graphical notation has been introduced to make the representation of the obtained results more readable. This consists of a table with three lines and a number of columns that depends on the number of displacement variables which are used in the expansion. All 15 terms of the expansion are reported in Table 1. The fourth-order model, $N = 4$, is related to the following expressions:

$$\begin{aligned} u_x &= u_{x_1} + z u_{x_2} + z^2 u_{x_3} + z^3 u_{x_4} + z^4 u_{x_5} \\ u_y &= u_{y_1} + z u_{y_2} + z^2 u_{y_3} + z^3 u_{y_4} + z^4 u_{y_5} \\ u_z &= u_{z_1} + z u_{z_2} + z^2 u_{z_3} + z^3 u_{z_4} + z^4 u_{z_5} \end{aligned} \tag{23}$$

The first column reports the constant terms, the second column the linear terms and so on up to the fifth column which is related to the fourth-order terms. White and black triangles are used to denote the inactive and active terms, respectively, Fig. 4 shows the case in which

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
		$N = 1$		
▲▲ ▲▲ ▲▲	100.0	100.0	66.7	2.1×10^5
△▲ ▲▲ ▲▲	100.0	100.0	66.7	2.1×10^5
▲▲ △▲ ▲▲	100.0	100.0	66.7	2.1×10^5
▲▲ ▲▲ △▲	9.3×10^{-6}	6.5×10^{-2}	0.0	50.0
▲△ ▲▲ ▲▲	0.1	3.2×10^{-2}	133.3	156.2
▲▲ ▲△ ▲▲	0.1	3.2×10^{-2}	4.4×10^{-2}	156.2
▲▲ ▲▲ ▲△	100.0	100.0	66.7	2.1×10^5
		$N = 2$		
▲▲▲ ▲▲▲ ▲▲▲	100.0	100.0	72.3	101.7
▲▲△ ▲▲▲ ▲▲▲	100.0	100.0	72.3	101.7
▲▲▲ ▲▲△ ▲▲▲	100.0	100.0	72.3	101.7
▲▲▲ ▲▲▲ ▲▲△	73.5	113.1	66.7	1.6×10^5
		$N = 3$		
▲▲▲▲ ▲▲▲▲ ▲▲▲▲	100.0	100.0	100.0	127.6
▲▲▲△ ▲▲▲▲ ▲▲▲▲	100.0	100.0	72.3	114.7
▲▲▲▲ ▲▲▲△ ▲▲▲▲	100.0	100.0	100.0	114.7
▲▲▲▲ ▲▲▲▲ ▲▲▲△	100.0	100.0	100.0	127.6

Fig. 3 Influence of each displacement variable on the solution for expansion orders lower than fourth. $a/h = 100$. Isotropic plate.

the parabolic term of the in-plane displacement of the expansion in the y-direction is discarded. The explicit displacement model related to Fig. 4 is

$$
\begin{aligned}
u_x &= u_{x_1} + z u_{x_2} + z^2 u_{x_3} + z^3 u_{x_4} + z^4 u_{x_5} \\
u_y &= u_{y_1} + z u_{y_2} + \qquad + z^3 u_{y_4} + z^4 u_{y_5} \\
u_z &= u_{z_1} + z u_{z_2} + z^2 u_{z_3} + z^3 u_{z_4} + z^4 u_{z_5}
\end{aligned} \tag{24}
$$

The elimination of a term, as well as the evaluation of its effectiveness in the analysis, can be obtained either by rearranging the rows and columns of the stiffness matrix or by exploiting a penalty technique (a further technique has been discussed in [59]). The corresponding results are compared with those given by a full fourth-order model using the percentage variations δ_u and δ_σ or $\bar{\delta}_u$ and $\bar{\delta}_\sigma$. These parameters are defined according to the following formulas:

Table 1 Locations of the displacement variables within the tables layout

$N=0$	$N=1$	$N=2$	$N=3$	$N=4$
u_{x_1}	$u_{x_2}\ z$	$u_{x_3}\ z^2$	$u_{x_4}\ z^3$	$u_{x_5}\ z^4$
u_{y_1}	$u_{y_2}\ z$	$u_{y_3}\ z^2$	$u_{y_4}\ z^3$	$u_{y_5}\ z^4$
u_{z_1}	$u_{z_2}\ z$	$u_{z_3}\ z^2$	$u_{z_4}\ z^3$	$u_{z_5}\ z^4$

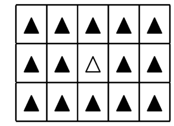


Fig. 4 Symbolic representation of the reduced kinematics model with u_{y_3} deactivated.

$$
\begin{aligned}
\delta_{u_z} &= \frac{u_z}{u_{z_{N=4}}} \times 100, \qquad \delta_{\sigma_{zz}} = \frac{\sigma_{zz}}{\sigma_{zz_{N=4}}} \times 100, \\
\bar{\delta}_{u_z} &= \left\| \frac{u_z - u_{z_{N=4}}}{u_{z_{N=4}}} \right\| \times 100, \qquad \bar{\delta}_{\sigma_{zz}} = \left\| \frac{\sigma_{zz} - \sigma_{z_{N=4}}}{\sigma_{zz_{N=4}}} \right\| \times 100
\end{aligned} \tag{25}
$$

Where subscript $N = 4$ denotes the values that correspond to the plate theory given by Eq. (23). Parameters related to other stress or displacement values could also be introduced (δ_{u_x}, $\bar{\delta}_{\sigma_{xx}}$, etc.).

It is important to notice that a displacement variable of the expansion in the displacement model is considered to be noneffective with respect to a specific output component when, if neglected (removed from the formulation), it does not introduce any changes in the results according to a fixed accuracy.

The accuracy is here fixed to be as 0.05%, that is, a term is considered negligible if the error caused by its absence in the kinematics model, $\bar{\delta}$, is smaller than 0.05%. When conducting the analysis of each displacement variable, a reduced kinematics model, if any exists, is established which is equivalent to a fourth-order expansion.

The numerical investigation has considered different plate layouts: 1) isotropic plates, 2) orthotropic plates, 3) cross-ply composite plates, and 4) multilayered plates made of two isotropic layers of different materials. Furthermore, the effects on the definition of the reduced model of the following geometrical/mechanical parameters have been evaluated: 1) length-to-thickness ratio (a/h), 2) orthotropic ratio (E_L/E_T), and 3) ply sequence.

IV. Results and Discussion

A simply-supported plate has been considered. The adopted reference frame is shown in Fig. 2. A bisinusoidal transverse distributed load is applied to the top surface:

$$
p_z = \bar{p}_z \sin\left(\frac{mx}{a}\right) \cos\left(\frac{ny}{b}\right) \tag{26}
$$

with $a = 0.1$ [m], b is assumed equal to a. $\bar{p}_z$ is the applied load amplitude, $\bar{p}_z = 1$ [kPa], and m, n are the wave number in the two in-plane, plate directions. Attention has been restricted to the case $m = n = 1$. Equation (26) is suitable for the Navier solution which admits harmonic loading distributions. If not differently specified, stresses are computed through Hooke's laws. u_z, σ_{xx} and σ_{zz} are computed at $[a/2, b/2, h/2]$, whereas σ_{xz} is computed at $[0, b/2, 0]$ and σ_{yz} at $[a/2, 0, 0]$. In these x-y locations, the chosen output variables are equal to the amplitude of the harmonic distributions.

A. Isotropic Plates

An isotropic plate has been considered first. Young's modulus, E, is equal to 73 GPa. Poisson's ratio, ν, is equal to 0.34. Four different length-to-thickness ratios, a/h, are considered: 100, 10, 5 and 2, that is, thin, moderately thick, thick and very thick plates, respectively.

Table 2 shows the comparison between the three-dimensional and fourth-order solutions. The three-dimensional exact elasticity results are obtained as in [60–62]. three-dimensional solutions are difficult to be obtained in the most general cases of geometry and boundary conditions; anyway these solutions play a fundamental role to assess classical and refined two-dimensional models. It should be noticed that the fourth-order model, $N = 4$, perfectly matches the exact solution, therefore, $N = 4$ has been chosen as the reference for the present analyses.

Table 2 Comparison of the solutions obtained via three-dimensional and a fourth-order models in case of isotropic plate

a/h	$\bar{u}_{z_{3D}}$ [a]	$\bar{u}_{z_{N=4}}$	$\bar{\sigma}_{xx_{3D}}$	$\bar{\sigma}_{xx_{N=4}}$	$\bar{\sigma}_{yz_{3D}}$	$\bar{\sigma}_{yz_{N=4}}$	$\bar{\sigma}_{zz_{3D}}$	$\bar{\sigma}_{zz_{N=4}}$
100	2.7248	2.7248	0.2037	0.2037	0.2387	0.2387	0.0100	0.0100
10	2.8345	2.8345	0.2068	0.2068	0.2383	0.2385	0.1000	0.1000
5	2.2056	2.2056	0.2168	0.2168	0.2371	0.2376	0.2000	0.2002
2	7.3826	7.3813	0.3145	0.3165	0.2277	0.2306	0.5000	0.5008

[a] $\bar{u}_z = (u_z 100 E_T h^3)/(\bar{p}_z a^4)$; $\bar{\sigma}_{xx} = (\sigma_{xx})/(\bar{p}_z (a/h)^2)$; $\bar{\sigma}_{yz} = (\sigma_{yz})/(\bar{p}_z (a/h))$; $\bar{\sigma}_{zz} = (\sigma_{zz})/(\bar{p}_z (a/h))$.

The results of the effectiveness of each displacement variable are given in Fig. 3. A thin plate geometry has been considered. Linear, parabolic and cubic expansions are analyzed whereas the fourth-order case is given in Fig. 5. The percentage variation, δ, introduced neglecting each displacement variable, is evaluated for u_z, σ_{xx}, σ_{xz} and σ_{zz}.

The first row in Fig. 3 refers to the full linear expansion, that is, 6 terms are retained in the CUF formulated theory. A thin plate geometry is first considered. A thickness locking correction for classical and refined theories has been made according to [56,57]. The corresponding columns show that, for a full $N = 1$ expansion, the following remarks can be made:

1) The transverse displacement and in-plane stress values have reached 100% of the reference solution ($N = 4$).

2) Only 66.7% of the transverse shear stress (computed via Hooke's law) is instead detected.

3) The transverse normal stress is completely wrong, even though a thin plate is under consideration.

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
△▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ △▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ △▲▲▲▲	1.3×10^{-5}	1.2×10^{-2}	1.3×10^{-2}	81.3
▲△▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	0.2	0.2	299.7	100.0
▲▲▲▲▲ ▲△▲▲▲ ▲▲▲▲▲	0.2	0.2	0.2	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲△▲▲▲	100.0	100.0	100.0	139.1
▲▲△▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲△▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲△▲▲	94.6	74.2	101.8	-8.1×10^4
▲▲▲△▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	72.3	100.0
▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲△▲	100.0	100.0	100.0	100.0
▲▲▲▲△ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲△ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲△	100.0	100.0	100.0	127.6

Fig. 5 Influence of each displacement variable of a fourth-order model on the solution. $a/h = 100$. Isotropic plate.

The further six case theories in which each single term is neglected are quoted in rows 2–7. The following comments can be made:

1) As could be expected, the membrane terms u_{x_1} and u_{y_1} do not have any influence on the results, and they can therefore be removed. The plate, in fact, shows symmetric deformation with respect to its reference surface.

2) The results are completely destroyed when the rotational variables u_{x_2} and u_{y_2} (which describe most of the bending and transverse shear deformation of the problem) are neglected.

3) Membrane transverse displacement cannot be neglected otherwise the plate would not bend.

4) The linear terms in the transverse displacement have no influence on the obtained results.

The full $N = 2$ case is given in row 8 and only three corresponding subcases are analyzed in rows 9–10–11. For the sake of brevity, all the constant and linear terms have been kept active (more complete analyses can be found in [63]). The quoted results make the following remarks evident:

1) The parabolic terms improve the transverse shear stresses (from 66.7 to 72.3%).

2) The transverse normal stress evaluation is almost completely related (101%) to the parabolic term u_{z3} in the transverse displacement expansion.

3) As for the constant terms in the previous cases, the parabolic terms u_{x_3} e u_{y_3} are meaningless, and they can be removed because the bending problem is symmetrical with respect to the middle plane.

Third-order plate theories are considered in the remaining rows. The full third-order case reaches 100% accuracy for transverse displacement, in-plane normal and transverse shear stresses, whereas the transverse normal stress is less accurate than the parabolic case. This phenomenon, which consists of oscillations of the Taylor-type series versus the three-dimensional solution in the evaluation of σ_{zz}, is known in CUF works [59]; it disappears, in fact, in the $N = 4$ case, see Fig. 5.

Sixteen fourth-order plate theories are considered in Fig. 5. The considered cases fully explore the effectiveness of 15 (3×5)

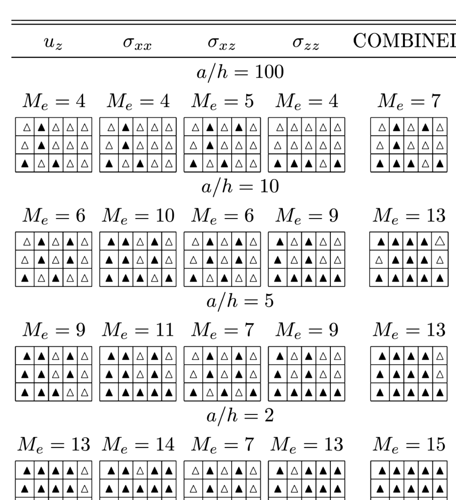


Fig. 6 Comparison of the sets of effective terms for isotropic plates with different a/h.

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
△▲△▲△ △▲△▲△ ▲△▲△△	100.0	100.3	100.0	77.6
▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	100.0	100.0	100.0	99.7
▲△▲△△ ▲△▲△△ ▲▲▲▲▲	4.9	1.7	66.6	100.0

Fig. 7 Comparison of different plate models in terms of accuracy of the solution. $a/h = 10$. Isotropic plate.

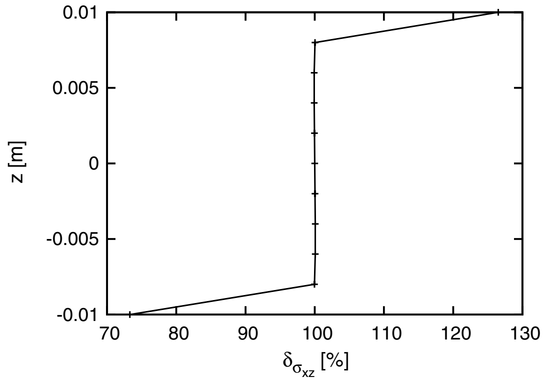


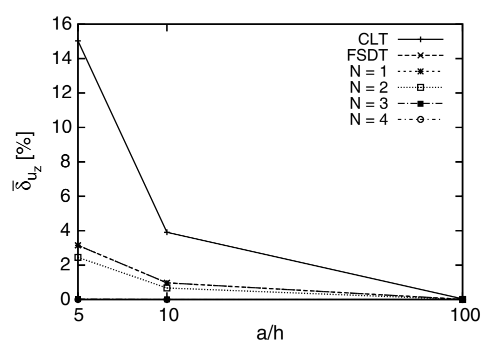
Fig. 8 $\delta_{\sigma_{xz}}$ vs z. $a/h = 5$. Isotropic plate. Inactive term: u_{x_5}.

displacement variables. For the sake of brevity, this study has not been documented in the $N = 1, 2, 3$ cases. The following comments can be made.

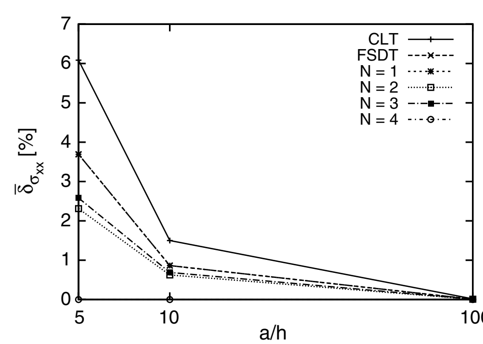
1) As in the error definition, the full $N = 4$ analysis does not show any error.

2) It has been confirmed that the constant term, u_{z_1}, and linear terms, u_{x_2} and u_{y_2}, are the most important ones to detect u_z, σ_{xx} and σ_{xz}, whereas u_{z_2} is the most important term for the transverse normal stress evaluation.

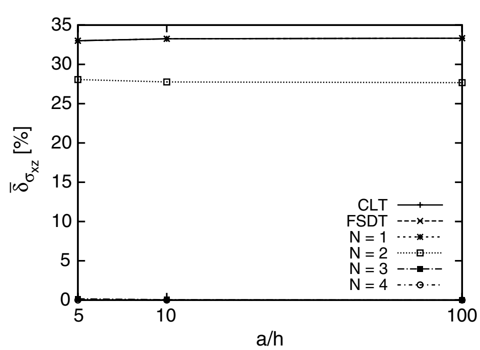
For the sake of brevity, the tables referring to different a/h values are not reported here, but they can be found in [63]. A rather quite comprehensive analysis is instead given in Fig. 6, which considers different plate geometries. The sets of effective terms are reported, that is, the plate models required to detect the fourth-order solution are shown. Each row refers to a fixed length-to-thickness ratio. Each column refers to an output variable. The last column gives the



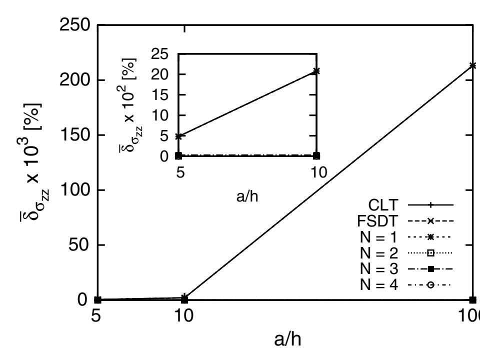
a) $\bar{\delta}_{u_z}$



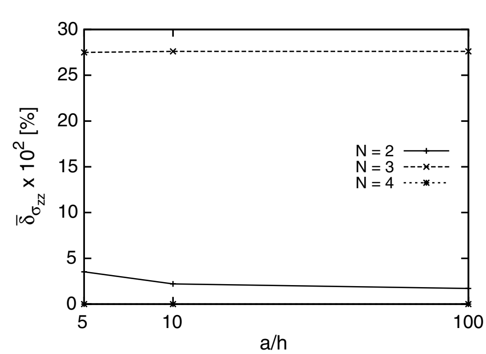
b) $\bar{\delta}_{u_{xx}}$



c) $\bar{\delta}_{\sigma_{xz}}$



d) $\bar{\delta}_{\sigma_{zz}}$



e) $\bar{\delta}_{\sigma_{zz}}$, $N > 1$

Fig. 9 $\bar{\delta}$ vs a/h. Isotropic plate.

Table 3 Comparison of the solutions obtained via three-dimensional and a fourth-order models in case of orthotropic plate

E_L/E_T	$\bar{u}_{z_{3D}}$ [a]	$\bar{u}_{z_{N=4}}$	$\bar{\sigma}_{xx_{3D}}$	$\bar{\sigma}_{xx_{N=4}}$	$\bar{\sigma}_{yz_{3D}}$	$\bar{\sigma}_{yz_{N=4}}$	$\bar{\sigma}_{zz_{3D}}$	$\bar{\sigma}_{zz_{N=4}}$
5	1.6914	1.6914	0.3983	0.3983	0.1254	0.1253	0.1000	0.1000
10	1.1559	1.1560	0.4843	0.4843	0.0831	0.0830	0.1000	0.1000
25	0.6970	0.6970	0.5827	0.5827	0.0472	0.0472	0.1000	0.1000
50	0.5085	0.5086	0.6637	0.6641	0.0326	0.0326	0.1000	0.1000
100	0.4038	0.4043	0.7758	0.7785	0.0245	0.0245	0.1000	0.1000

[a] $\bar{u}_z = (u_z 100 E_T h^3)/(\bar{p}_z a^4)$; $\bar{\sigma}_{xx} = (\sigma_{xx})/(\bar{p}_z(a/h)^2)$; $\bar{\sigma}_{yz} = (\sigma_{yz})/(\bar{p}_z(a/h))$; $\bar{\sigma}_{zz} = (\sigma_{zz})/(\bar{p}_z(a/h))$.

expansion terms needed to exactly detect the whole considered outputs exactly. M_e states the number of terms (i.e. computational costs) of the theory necessary to meet the fourth-order accuracy requirements. The required terms are again those corresponding to the black triangles. Some remarks can be made:

1) As a/h increases, the theories become more computationally expensive (M_e increases).

2) Different choices of displacement variables are required to obtain exact different outputs.

3) The sets of effective terms differ to a great extent with an increase of a/h.

4) All 15 terms are necessary for very thick plate geometries.

Figure 6 is, as in the paper title, an attempt to offer both guidelines and recommendations for building plate theories for the considered plate problems. For instance, to accurately describe the transverse shear stress σ_{xz}, in the case of a/h equal to 10, six terms are needed. The corresponding kinematics model becomes

$$\begin{aligned} u_x &= z u_{x_2} + z^3 u_{x_4} \\ u_y &= z u_{y_2} + z^3 u_{y_4} \\ u_z &= u_{z_1} + z^2 u_{z_3} \end{aligned} \tag{27}$$

In other words, the latter table answers the following fundamental questions: for a given plate (isotropic, thin, simply-supported, transversally loaded, etc.), what are the terms that have to be retained (required) to obtain a fixed accuracy? How do these terms change with a change in the output variables ($u_z, \sigma_{xx}, \sigma_{xz}, \sigma_{zz}$)?

As a further example, Fig. 7 has been built drawing on Fig. 6. Three plate models are considered for $a/h = 10$. The first row refers to the plate theory needed to accurately detect u_z and σ_{xz}, whereas the second and third rows are related to σ_{xx} and σ_{zz}, respectively. The exact evaluation of transverse normal stress requires a set of terms which are not able to detect u_z, σ_{xx} and σ_{xz}.

The accuracy of a plate theory could also depend on the position where an output variable is evaluated in the thickness direction of the plate. As an example, Fig. 8 shows the error in the thickness direction for a given geometry and plate theory. The plate model consists of a fourth-order expansion with the u_{x_5} displacement variable deactivated:

$$\begin{aligned} u_x &= u_{x_1} + z u_{x_2} + z^2 u_{x_3} + z^3 u_{x_4} \\ u_y &= u_{y_1} + z u_{y_2} + z^2 u_{y_3} + z^3 u_{y_4} + z^4 u_{y_5} \\ u_z &= u_{z_1} + z u_{z_2} + z^2 u_{z_3} + z^3 u_{z_4} + z^4 u_{z_5} \end{aligned} \tag{28}$$

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta^*_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{yz}}$ [%]	$\delta^*_{\sigma_{yz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0	100.0	100.0	100.0
△▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.1	100.0	99.6	100.0	100.0	100.0
▲▲▲▲▲ △▲▲▲▲ ▲▲▲▲▲	100.0	99.9	100.0	100.0	100.0	92.0	100.1
▲▲▲▲▲ ▲▲▲▲▲ △▲▲▲▲	1.1	6.9×10^{-1}	2.2×10^{-1}	2.5×10^{-1}	5.8	6.8	81.4
▲△▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	97.1	92.0	111.1	70.0	94.7	94.6	100.0
▲▲▲▲▲ ▲△▲▲▲ ▲▲▲▲▲	61.4	61.4	60.7	60.7	1219.6	26.8	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲△▲▲▲	99.7	99.9	100.0	100.0	100.0	100.0	134.2
▲▲△▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲△▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0	100.0	100.0	100.1
▲▲▲▲▲ ▲▲▲▲▲ ▲▲△▲▲	100.0	100.1	100.0	100.0	99.9	99.7	105.0
▲▲▲△▲ ▲▲▲▲▲ ▲▲▲▲▲	90.2	74.0	70.6	110.4	91.8	91.9	100.0
▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0	67.6	100.6	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲△▲	100.0	100.0	100.0	100.0	100.0	100.0	100.1
▲▲▲▲△ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲△ ▲▲▲▲▲	100.0	100.0	100.0	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲△	100.0	100.0	100.0	100.0	100.0	99.7	115.7

(*): Computed through indefinite equilibrium equation

Fig. 10 Influence of each displacement variable of a fourth-order model on the solution. Orthotropic plate. $E_L/E_T = 100$.

u_z	σ_{xx}	σ_{xz}	σ_{zz}	COMBINED
$E_L/E_T = 5$				
$M_e = 7$	$M_e = 10$	$M_e = 6$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲▲▲△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△▲△ ▲△▲△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ ▲▲▲▲△ ▲▲▲▲▲
$E_L/E_T = 10$				
$M_e = 7$	$M_e = 10$	$M_e = 6$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲▲▲△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△▲△ ▲△▲△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ ▲▲▲▲△ ▲▲▲▲▲
$E_L/E_T = 25$				
$M_e = 6$	$M_e = 10$	$M_e = 4$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲▲△△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲△△△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ ▲▲▲▲△ ▲▲▲▲▲
$E_L/E_T = 50$				
$M_e = 5$	$M_e = 8$	$M_e = 4$	$M_e = 8$	$M_e = 11$
△▲△▲△ △▲△△△ ▲▲△△△	▲▲△▲△ ▲▲△△△ ▲▲△△▲	△▲△▲△ △▲△△△ ▲△△△△	▲△△△△ ▲△▲△△ ▲▲▲▲▲	▲▲△▲△ ▲▲▲△△ ▲▲▲▲▲
$E_L/E_T = 100$				
$M_e = 5$	$M_e = 6$	$M_e = 3$	$M_e = 7$	$M_e = 11$
△▲△▲△ △▲△△△ ▲▲△△△	▲▲△△△ ▲▲△△△ ▲▲△△△	△▲△△△ △▲△△△ ▲△△△△	△△△△△ ▲△▲△△ ▲▲▲▲▲	▲▲△▲△ ▲▲▲△△ ▲▲▲▲▲

Fig. 11 Comparison of the sets of effective terms for orthotropic plates with different E_L/E_T.

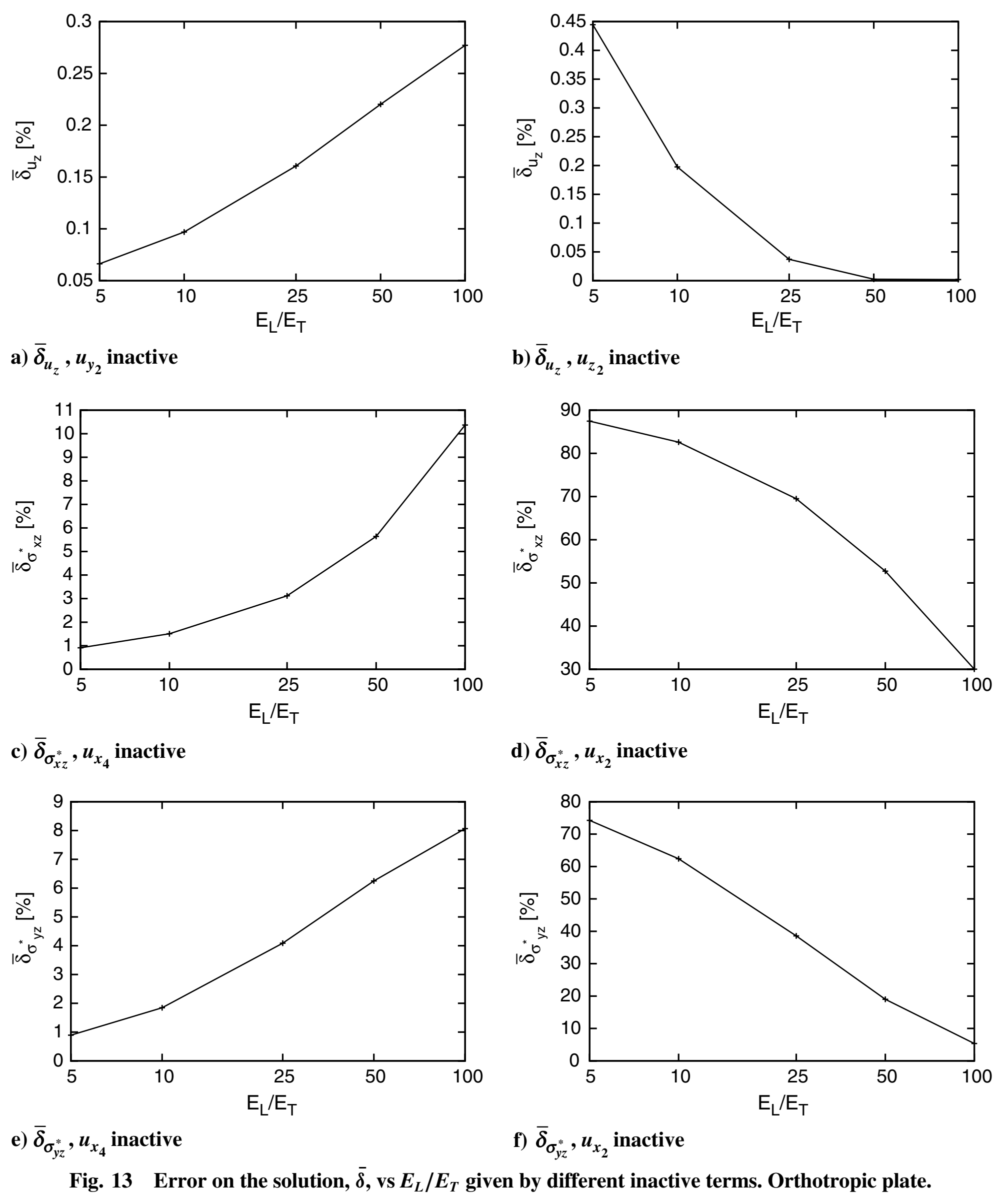
M_e/M	σ_{xz}	M_e/M	σ^*_{xz}	M_e/M	σ_{yz}	M_e/M	σ^*_{yz}
			$E_L/E_T = 5$				
6/15	△▲△▲△ △▲△▲△ ▲△▲△△	7/15	▲▲△▲△ △▲△▲△ ▲△▲△△	6/15	△▲△▲△ △▲△▲△ ▲△▲△△	8/15	△▲△▲△ ▲▲△▲△ ▲△▲△▲
			$E_L/E_T = 10$				
6/15	△▲△▲△ △▲△▲△ ▲△▲△△	7/15	▲▲△▲△ △▲△▲△ ▲△▲△△	6/15	△▲△▲△ △▲△▲△ ▲△▲△△	8/15	△▲△▲△ ▲▲△▲△ ▲△▲△▲
			$E_L/E_T = 25$				
4/15	△▲△▲△ △▲△△△ ▲△△△△	5/15	▲▲△▲△ △▲△△△ ▲△△△△	6/15	△▲△▲△ △▲△▲△ ▲△▲△△	8/15	△▲△▲△ ▲▲△▲△ ▲△▲△▲
			$E_L/E_T = 50$				
4/15	△▲△▲△ △▲△△△ ▲△△△△	5/15	▲▲△▲△ △▲△△△ ▲△△△△	6/15	△▲△▲△ △▲△▲△ ▲△▲△△	8/15	△▲△▲△ ▲▲△▲△ ▲△▲△▲
			$E_L/E_T = 100$				
3/15	△▲△△△ △▲△△△ ▲△△△△	5/15	▲▲△▲△ △▲△△△ ▲△△△△	6/15	△▲△▲△ △▲△▲△ ▲△▲△△	8/15	△▲△▲△ ▲▲△▲△ ▲△▲△▲

(∗): Computed through indefinite equilibrium equation

Fig. 12 Comparison of the sets of effective terms for orthotropic plates with different shear stress computation methods.

The difficulty of the plate theory to satisfy well-known homogeneous top/bottom conditions on transverse shear stresses, which leads to an increasing error at the top/bottom locations, is evident. Figure 9 shows, in graphical form, the errors $\bar{\delta}$, in terms of plate-thickness ratio. This diagram consists of a typical result of asymptotic-type formulated theories. Classical CLT and FSDT analyses are reported for comparison purposes. It has been confirmed that classical models are not able to accurately trace transverse shear and normal stress variables (computed by Hooke's law) even though thin plate geometries are considered. The analysis of the results related to CLT, FSDT and $N = 4$ confirms that CLT consists of a first approximation of exact solution in thin plate geometries. On the other hand, FSDT analysis, as it is known because Koiter's recommendation [64] (which states that the transverse shear and normal stresses have to be taken into account at the same time to refine models for isotropic plates/shells) shows difficulties to preserve the same accuracy for various problems in the thin plate cases. More recent work on that subject for the case of laminated plates, shells and beams are those by Carrera [65,66], and Song and Waas [67]. Quite different responses are obtained if different outputs are considered. a/h remains a fundamental parameter to establish the accuracy of a given plate theory.

In conclusion, the analysis of an isotropic plate suggests the following main guidelines and recommendations:



a) $\bar{\delta}_{u_z}$, u_{y_2} **inactive**

b) $\bar{\delta}_{u_z}$, u_{z_2} **inactive**

c) $\bar{\delta}_{\sigma^*_{xz}}$, u_{x_4} **inactive**

d) $\bar{\delta}_{\sigma^*_{xz}}$, u_{x_2} **inactive**

e) $\bar{\delta}_{\sigma^*_{yz}}$, u_{x_4} **inactive**

f) $\bar{\delta}_{\sigma^*_{yz}}$, u_{x_2} **inactive**

Fig. 13 Error on the solution, $\bar{\delta}$, vs E_L/E_T given by different inactive terms. Orthotropic plate.

1) Accurate evaluations of σ_{zz} require the use of $u_{z_1}, \ldots, u_{z_5}$ variables, whereas accurate evaluations of u_z, σ_{xx} and σ_{xz} require the variables u_{z_1}, u_{x_2}, u_{y_2} and u_{z_3}.

2) Different plate theories need to be constructed if the evaluated outputs change (the case of σ_{zz} is quite extreme) or the plate geometry varies.

3) The computational cost of plate theories decrease as the accuracy or thickness decreases.

4) The error depends on the thickness coordinate, z; in particular, simplified theories experience difficulties in fulfilling boundary conditions for transverse shear stress.

5) It has been confirmed that the use of CLT and FSDT yields good results in detecting u_z and σ_{xx} in the case of thin plates, whereas these theories are not able to properly predict σ_{xz} and σ_{zz} in either the case of thin or thick plates.

B. Orthotropic Plate

Orthotropic plates have been considered to assess the accuracy of the plate theory vs orthotropic ratio, E_L/E_T. It is a well-known fact that orthotropic plates, such as laminated composite structures,

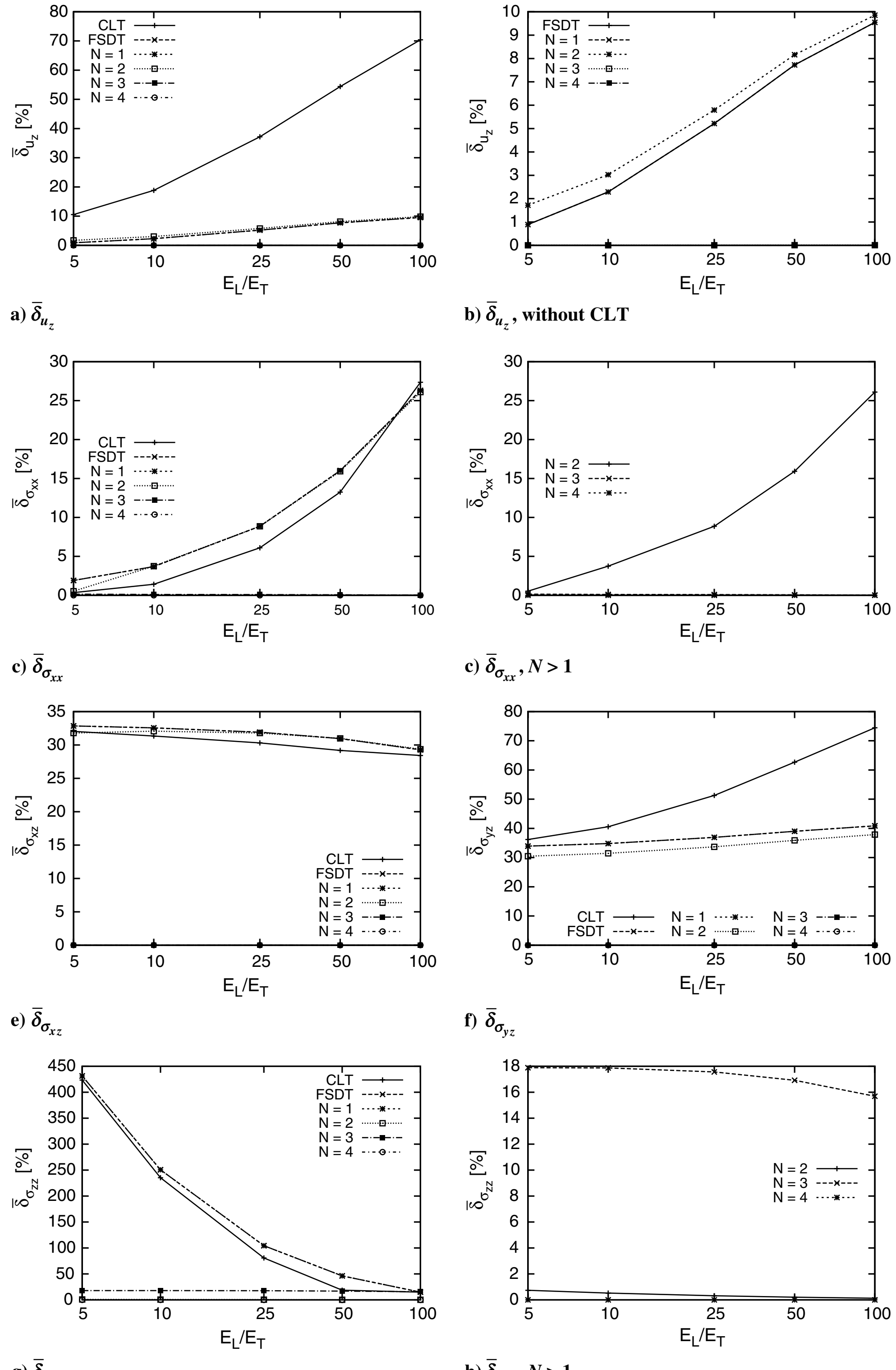


Fig. 14 $\bar{\delta}$ vs E_L/E_T. Orthotropic plate.

Table 4 Reference fourth-order solutions obtained in case of composite plate

Ply sequence	$\bar{u}_z$[a]	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{zz}$
0°/0°/0°	0.3022	0.5674	0.4509	0.0300
0°/90°/0°	0.2989	0.5647	0.4517	0.0299
0°/0°/90°	0.6681	0.0468	0.3455	0.0407

[a] $\bar{u}_z = (u_z 100 E_T h^3)/(\bar{p}_z a^4)$; $\bar{\sigma}_{xx} = (\sigma_{xx})/(\bar{p}_z (a/h)^2)$; $\bar{\sigma}_{yz} = (\sigma_{yz})/(\bar{p}_z (a/h))$; $\bar{\sigma}_{zz} = (\sigma_{zz})/(\bar{p}_z (a/h))$.

exhibit larger shear deformations than metallic structures made of isotropic materials.

The analysis of such plates is therefore of particular interest for the present investigation.

Young's modulus along the transverse direction, E_T, is assumed as high as 1 GPa. Different orthotropic ratios, E_L/E_T, are assumed: 5, 10, 25, 50, and 100, where E_L stands for Young's modulus along the longitudinal direction. Young's thickness modulus, E_z, is assumed equal to E_T. The shear moduli are assumed as high as 0.39 GPa. Poisson's ratios, ν_{LT} and ν_{Lz}, are equal to 0.25, and a/h is assumed equal to 10.

Table 3 shows a comparison between the three-dimensional and fourth-order solutions. Three-dimensional exact elasticity results are obtained as in [60,61,68]. Because the fourth-order model perfectly matches the exact solution, it has once again been chosen as the reference solution for the present analyses.

Figure 10 evaluates the effectiveness of 15 (3 × 5) displacement variables in detecting different output variables in the case of $E_L/E_T = 100$. Transverse shear stresses are computed via both Hooke's law and indefinite equilibrium equations. For the sake of brevity, the tables that refer to different E_L/E_T and a/h values are not reported here, but can be found in [63]. Figure 10 highlights which terms are effective in detecting an output variable to obtain a fourth-order model accuracy. The sets of effective variables for different orthotropic ratios are summarized in Figs. 11 and 12.

The error on u_z and shear stresses due to the deactivation of different terms are plotted in Fig. 13 for different orthotropic ratios. Shear stresses are computed via indefinite equilibrium equations. The error given for different plate models is plotted in Fig. 14 for different orthotropic ratios. Classical (CLT and FSDT), linear and higher-order theories are considered.

The obtained results suggest the following comments:

1) As in the isotropic case, each displacement variable affects a displacement/stress output with a different magnitude, for instance, the absence of u_{z_2} heavily corrupts the σ_{zz} results, whereas it has no influence on the other considered outputs.

2) The most important terms for the evaluation of σ_{zz} are $u_{z_1}, \ldots, u_{z_5}$, whereas u_z, σ_{xx} and σ_{xz} require u_{z_1}, u_{x_2} and u_{y_2}.

3) The construction of a plate theory that is able to evaluate a particular output variable is strongly influenced by the orthotropic ratio.

4) The plate theory needed to furnish an accurate description of several outputs tends to have lower M_e for larger E_L/E_T values (as shown in the last column of Fig. 11).

5) The orthotropic ratio, E_L/E_T, plays a similar role as the length-to-thickness ratio, a/h; these two parameters are the most significant in evaluating the accuracy of a given plate theory.

6) It is of interest to consider the effect of a high orthotropic ratio, $E_L/E_T = 100$, on the influence of u_{y_2} on σ_{yz}. The absence of u_{y_2} makes the computation of σ_{yz} totally wrong ($\delta_{\sigma_{yz}} = 1219.6\%$). Additional results on these detrimental effect has been detailed in [63].

C. Cross-Ply Composite Plate

Composite plates have been analyzed to assess the plate theory accuracy vs stacking sequence. Attention is restricted to higher-order theories, as in Eq. (3). The authors are aware that laminated structures require more adequate descriptions, such as those given by the so-called zig-zag theories as well as a LW description. Such analyses are herein omitted but could be the subject of future investigations. The readers are addressed to the already mentioned review papers as well as to the historical review on zig-zag theories in [69].

A three-layer composite plate has been analyzed. E_L is equal to 40 GPa. E_T and E_z are equal to 1 GPa. ν_{LT} and ν_{Lz} are equal to 0.5 and 0.6, respectively. Each layer is 0.001 [m] thick. Three stacking sequences are considered: two symmetrical 0°/90°/0° and 0°/0°/0°, and one asymmetrical 0°/0°/90°. Table 4 shows the fourth-order reference solutions.

Figure 15 presents the effectiveness of all 15 displacement variables in the case of the 0°/90°/0° stacking sequence. For the sake

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
△▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ △▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ △▲▲▲▲	1.2×10^{-2}	1.0×10^{-2}	3.7×10^{-2}	81.5
▲△▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	11.8	6.9	163.7	99.7
▲▲▲▲▲ ▲△▲▲▲ ▲▲▲▲▲	11.5	11.5	11.4	100.9
▲▲▲▲▲ ▲▲▲▲▲ ▲△▲▲▲	100.0	100.0	100.0	133.6
▲▲△▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲△▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲△▲▲	99.9	99.7	100.0	−249.5
▲▲▲△▲ ▲▲▲▲▲ ▲▲▲▲▲	99.4	99.2	72.6	100.1
▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲△▲	100.0	100.0	100.0	100.0
▲▲▲▲△ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲△ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲△	100.0	100.0	100.0	115.4

Fig. 15 Influence of each displacement variable of a fourth-order model on the solution. Composite plate: 0°/90°/0°.

u_z	σ_{xx}	σ_{xz}	σ_{zz}	COMBINED
0°/0°/0°				
$M_e = 5$	$M_e = 5$	$M_e = 4$	$M_e = 4$	$M_e = 7$
△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△△△△	△△△△△ △△△△△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲▲▲△▲
0°/90°/0°				
$M_e = 5$	$M_e = 5$	$M_e = 4$	$M_e = 7$	$M_e = 7$
△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△△△△	△▲△▲△ △▲△△△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲▲▲△▲
0°/0°/90°				
$M_e = 9$	$M_e = 12$	$M_e = 10$	$M_e = 14$	$M_e = 14$
▲▲▲▲▲ ▲▲△△△ ▲△▲△△	▲▲▲▲▲ ▲▲△△△ ▲▲▲▲▲	▲▲▲▲▲ ▲▲△△△ ▲▲▲△△	▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲

Fig. 16 Comparison of the sets of effective terms for composite plates with different stacking sequences.

Table 5 Reference fourth-order solutions obtained in case of bimetallic plate

$\bar{u}_z$[a]	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yz}$
3.4925	0.2261	0.1893

[a] $\bar{u}_z = (u_z 100 E_T h^3)/(\bar{p}_z a^4)$; $\bar{\sigma}_{xx} = (\sigma_{xx})/(\bar{p}_z (a/h)^2)$; $\bar{\sigma}_{yz} = (\sigma_{yz})/(\bar{p}_z (a/h))$.

of brevity, the tables which refer to the other two stacking layouts are not reported here, but they can be found in [63].

Figure 16 shows the plate model for each stacking sequence and output variable as well as for the combined evaluation of u_z, σ_{xx}, σ_{xz} and σ_{zz} to obtain a fourth-order model accuracy.

The analysis of the cross-ply composite plate highlights the following main aspects related to the choice of the plate theory:

1) The stacking sequence influences the construction of adequate plate models to a great extent; it plays a similar role as the geometry and the orthotropic ratio.

	δ_{u_z} [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0
△▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	99.1	106.8	99.9
▲▲▲▲▲ △▲▲▲▲ ▲▲▲▲▲	99.1	97.5	99.1
▲▲▲▲▲ ▲▲▲▲▲ △▲▲▲▲	1.8×10^{-4}	3.3×10^{-2}	3.9×10^{-2}
▲△▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	0.9	0.7	302.1
▲▲▲▲▲ ▲△▲▲▲ ▲▲▲▲▲	0.9	0.8	0.7
▲▲▲▲▲ ▲▲▲▲▲ ▲△▲▲▲	99.5	103.2	99.8
▲▲△▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	97.3
▲▲▲▲▲ ▲▲△▲▲ ▲▲▲▲▲	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲△▲▲	95.3	77.6	101.4
▲▲▲△▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	72.9
▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲△▲	100.0	99.5	100.0
▲▲▲▲△ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	98.1
▲▲▲▲▲ ▲▲▲▲△ ▲▲▲▲▲	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲△	100.0	99.9	100.0

Fig. 17 Influence of each displacement variable of a fourth-order model on the solution. Al-Ti composite plate.

u_z	σ_{xx}	σ_{xz}	COMBINED
$M_e = 7$	$M_e = 9$	$M_e = 10$	$M_e = 12$
▲▲△△△ ▲▲△△△ ▲▲▲△△	▲▲△△△ ▲▲△△△ ▲▲▲▲▲	▲▲▲▲▲ ▲▲△△△ ▲▲▲△△	▲▲▲▲▲ ▲▲△△△ ▲▲▲▲▲

Fig. 18 Reduced plate models equivalent to a fourth-order expansion in case of Al-Ti composite plate.

2) An asymmetric lamination sequence requires a considerably higher number of displacement variables than a symmetric one.

D. Bimetallic Plate

A composite plate made of two isotropic layers has been considered. Al-5086 and Ti-22 are used. The former has E equal to 70.3 GPa and ν equal to 0.33. The latter has E equal to 110 GPa and ν equal to 0.34. Each layer is 0.001 m thick. Table 5 shows the fourth-order reference solutions.

Figure 17 reports the effectiveness of each displacement variable for three different outputs. Figure 18 shows the displacement variables that have to be retained to obtain a fourth-order model accuracy.

The following remarks can be made:

1) Constant and linear terms are always needed, whereas the choice of the higher-order terms depends on the considered output variable.

2) The total number of retained terms is larger than that required for an isotropic plate with similar a/h values.

V. Conclusions

The effectiveness of each displacement variable of higher-order plate theories has been investigated in this paper. The CUF has been used for the systematic implementation of refined models. Navier-type closed-form solutions have been adopted for the analyses. Isotropic, orthotropic and composite plates have been considered. Thin and thick structures have been analyzed. Simply-supported boundary conditions have been considered. The plates have been subjected to bisinusoidal transverse pressure loadings. Full and reduced models have been examined to evaluate the role of each problem variable with respect to a fourth-order solution. The role of each displacement variable has been described in terms of displacement and stress components, referring to a fourth-order model solution. The contribution of each term to the accuracy of the solution has been evaluated, introducing the so-called mixed axiomatic/asymptotic method, which is able to recognize the effectiveness of each displacement variable of an arbitrary refined plate theory.

It can be stated that the choice of the plate model which suits the accuracy requirements for a given problem is dominated by 1) the length-to-thickness ratio, 2) the orthotropic ratio, and 3) the lamination sequence.

It has also been found that each displacement/stress component would require its own plate model (which differs according to the change in outputs) to obtain exact results. Moreover, the accuracy of the solution also depends on the thickness coordinate z.

The following points need to be mentioned:

1) The number of retained variables is very closely related to the geometrical/mechanical configuration of the considered problem.

2) Remarkable benefits, in terms of total amount of problem variables, are obtained for thin plates or for symmetrical laminations.

3) The use of full models is mandatory when a complete set of results is needed.

CUF has shown to be well able to deal with a method that could be stated as a mixed assiomatic/asymptotic structural analysis of different plate structures. Its usage has made it possible to consider the plate theory as well as the structural configuration as free parameters of the analysis. Two main benefits can be obtained using CUF:

1) It permits the accuracy of each problem variable to be evaluated by comparing the results with more detailed analyses (also provided by CUF); no mathematical/variational techniques are needed as in the case of asymptotic-type analyses.

2) It offers the possibility of considering the accuracy of the results as an input, whereas the output is represented by the set of displacement variables which are able to fulfill the requirement.

Future investigations could consider shell geometries, dynamics, multifield problems, finite element analysis and nonlinear problems.

Appendix: Example of Coded Statements

The nine components of the fundamental nucleus of the stiffness matrix are here reported as they are stated in the FORTRAN code. They represent the coded translations of the nine statements of Eq. (22):

$$K(1,1) = \mathrm{ETS}(k,\tau,s) * \mathrm{QPP}(k,1,1) * \mathrm{ALPHA} * *2 + \mathrm{ETS}(k,\tau,s) * \mathrm{QPP}(k,6,6) * \mathrm{BETA} * *2 + \mathrm{ETZSZ}(k,\tau,s) * \mathrm{QPP}(k,5,5) \tag{A1a}$$

$$K(1,2) = -\mathrm{ETS}(k,\tau,s) * (-\mathrm{ALPHA} * \mathrm{BETA} * (\mathrm{QPP}(k,1,2) + \mathrm{QPP}(k,6,6))) \tag{A1b}$$

$$K(1,3) = -\mathrm{ALPHA} * (\mathrm{ETSZHB}(k,\tau,s) * \mathrm{QPP}(k,1,3) + \mathrm{ETZS}(k,\tau,s) * \mathrm{ALPHA} * \mathrm{QPP}(k,5,5) \tag{A1c}$$

$$K(2,1) = k(1,2) \tag{A1d}$$

$$K(2,2) = \mathrm{ETS}(k,\tau,s) * \mathrm{QPP}(k,2,2) * \mathrm{BETA} * *2 + \mathrm{ETS}(k,\tau,s) * \mathrm{QPP}(k,6,6) * \mathrm{ALPHA} * *2 + \mathrm{ETZSZ}(k,\tau,s) * \mathrm{QPP}(k,4,4) \tag{A1e}$$

$$K(2,3) = -\mathrm{BETA} * (\mathrm{ETSZ}(k,\tau,s) * \mathrm{QPP}(k,2,3) + \mathrm{ETZS}(k,\tau,s) * \mathrm{BETA} * \mathrm{QPP}(k,4,4) \tag{A1f}$$

$$K(3,1) = -\mathrm{ALPHA} * (\mathrm{ETZS}(k,\tau,s) * \mathrm{QNP}(k,1,3)) + \mathrm{ETSZ}(k,\tau,s) * \mathrm{ALPHA} * \mathrm{QPP}(k,5,5) \tag{A1g}$$

$$K(3,2) = -\mathrm{ETZS}(k,\tau,s) * \mathrm{BETA} * \mathrm{QPP}(k,2,3) + \mathrm{ETSZ}(k,\tau,s) * \mathrm{BETA} * \mathrm{QPP}(k,4,4) \tag{A1h}$$

$$K(3,3) = \mathrm{ETZSZ}(k,\tau,s) * \mathrm{QPP}(k,3,3) + \mathrm{ETS}(k,\tau,s) * \mathrm{ALPHA} * *2 * \mathrm{QPP}(k,5,5) + \mathrm{ETS}(k,\tau,s) * \mathrm{BETA} * *2 * \mathrm{QPP}(k,4,4) \tag{A1i}$$

The 3×3 elements of the nucleus matrix are denoted by the array K. $ET\ldots$ type arrays denote the line integrals on z corresponding to those in Eq. (14). These are affected by τ and s indexes which come from the product of the used expansion in z, $F_\tau F_s$. Q are the material properties as stated in Eq. (10). k denotes the layer number. By putting the preceding 3×3 elements in appropriate loops with indexes τ, s and k, stiffness matrices are obtained.

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